Derivatives and Integrals

Differentiation:

We previous studies, we defined the slope of the curve at a point as the limit of secant slopes, and this limit called derivative measure rate of which a function changes, and it is one of the most important ideas in calculus.

Derivatives are used to calculate velocity and acceleration, to estimate the rate of spread of a distance, to set of production to find the best dimensions of figures (cylindrical, circle …etc), and for many applications.

In this chapter, we develop techniques to calculate derivatives easily and learn how to use derivatives to approximate complicated functions.

The derivative of the function respect to the variable x is the function whose value at x is

We say that is differentiable (has derivative) at x if exists at every point in the domain of .

The slope of tangent line:

The slope of the curve at the point is the number

The tangent line to the curve at is the line through with this slope.

In the previous studies the slope

We say that is differentiable (has derivative) at if exists at every point in the domain of , then we call differentiable.

But if is not differentiable at every point in the domain of , for example sqort or rotational functions, then we write , then and approaches 0 if approaches .

Therefore, an equivalent definition of the derivative is as follows

Example1/ Derivative of square root function:

a- find the derivative of for

b- Find the tangent line equation to the curve at x=4

Solution:

a-

b-

The slope of the curve at is , then

The tangent line through the point , and

Then , then the point

The equation of tangent line through the point and the slope equal

In another method:

Then point

The equation of tangent line through the point and the slope equal

<u>Differentiable on an interval; one sided derivatives:</u>

A function is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.

Right-hand derivative at a

Left-hand derivative at b

Example 2/ Function is not differentiable at the origin, show that the function is differentiable on and but has no derivative at .

Solution:

To the right of the origin

To the left of the origin

Example3/ Show that the function is not differentiable at

Solution:

We apply the definition to examine if the derivative exists at

Example4/ Find derivative functions and values using the definition, calculate the derivatives of the functions, then find the values of the derivatives as specified.

1- , , ,

Solution:

2- , , ,

Solution:

Example5/ Find the derivative of the following functions:

1-

2-

3-

4-

Example6/ Differentiable the following functions and find the slope of the tangent line at the given value of the independent variable, and find the equation of tangent line.

1-

Solution:

The equation of tangent line , then we need

2-

3-

Example7/ Find equation of the tangent line at the indicated point for the following function:

Solution:

The slope at

2-

Solution:

Example8/ Find the values of the following derivatives:

1-

Solution:

2-

Solution:

3-

Solution:

<u>Differentiation Rules</u>:

We can differentiate functions without having to apply the definition of the derivative each time.

Powers, multiples, sums and differences, derivative product rule, derivative quotient rule, power rule for negative integers, second and higher order derivative.

1- **The first rule of every constant function is zero.**

If has the constant value then

2- **Power rule for positive integers,** if is a positive integer, then

3- **Constant multiple rule**

If u is a differentiable function of , and is a constant, then

4- **Derivative sum rule**, if and are differentiable functions of , then their sum is differentiable at every point where and are to be differentiable at each point.

And for difference rule

5- **Derivative product rule**, if and are differentiable at, then

6- **Derivative quotient rule**, if and are differentiable at, and if

Example9/

7- **Power rule for negative integers**, if is a negative integer and then

Example 10/

8- **Second and higher order derivatives**

If is a differentiable function, then its derivative is also a function. If is also differentiable, then we can differentiable to get a new function of denoted by . called second derivative, third derivative,…….,etc.

Example 11/

Example 12/ Finding higher derivatives for the function

Example13/ Find the first and second derivative for the following functions:

1-

2-

3-

Example14/ How a circles area changes with its diameter.

Solution: the area of a circle is related to its diameter by the equation

How fast does the area change with respect to the diameter when the diameter is 10 meter?

Solution:

When

Motion along a line: displacement, velocity, speed and acceleration:

Suppose that object is moving along a coordinate line (say an s-axis), so that we know its position on that line as a function of time:

The displacement of the object over the time interval from to

And average velocity of the object over that time interval is

To find the body's velocity at the exact instant , we take the limit of the average velocity over the interval from to as shrinks to Zero. This limit is the derivative of with the respect to .

Velocity (instantaneous velocity) is the derivative of position with respect to time. If body's position at time , then the bodys velocity at time is

Speed is the absolute value of velocity

Acceleration is the derivative of velocity with respect to time

Example 15/ Free fall of a heavy ball released from rest at time , where the equation of free fall is .

a- How many meters does the ball fall in first ?

b- What are the velocity, speed and acceleration?

Solution:

a- during the first , the ball falls

b- at any time ,

Derivatives of Trigonometric functions:

Example 16/ Prove that

Solution:

Example 17/ Derivative the following trigonometric functions:

Solutions:

1-

2-

3-

Example 18/ Find second derivative for the function ?

Solution:

Derivative of a composition function:

The derivative of the composite function is the derivative of times the derivative of called (Chain Rule), if is differentiable at the point is differentiable at . Then the composite function is differentiable at , and

Example 19/ The function is the composite of the functions . How are the derivatives of these functions related?

Solution:

We have

Since

Example 20/ The function is the composite of . Calculate derivatives?

Solution:

By substituting equations 2 and 3 in equation 1 , then

Example 21/ An object moves along the so that its position at any time is given by . Find the velocity of the object as a function of .

Solution:

The velocity

Method 1/

Method 2/

Example 22/ Derivative   with respect to .

Solution:

Example 23/ Find the derivative of .

Solution:

The chain rule with powers of a function:

If is a differentiable function of   and if is a differentiable function of , then substituting into the chain rule formula

If is a positive or negative integer and

Example 24/

Example 25/

Example 26/

Example 27/ If , find the value of .

Solution:

By substituting equation 2 in equation 1 , then

Antiderivative and Integrals:

A function is an antiderivative of on an interval if

Example 28/

Antiderivative formulas:

No.	Function	General antiderivative	Notes
1			
2			
3			
4			
5			
6			
7			

Example 29/ Find the general antiderivative of

Solution:

Note/ The function is not function has derivative , also the function has derivative , therefore, we can write , where represent arbitrary constant, therefore the two functions different by constant .

Example 30/

Solution:

Integrals:

A special symbol is used to denote the collection of all antiderivatives of a function .

Definition: The set of all antiderivatives of is the **indefinite integral** of with respect to x ,

Example 31/ Indefinite integration done by term and rewriting the constant of integration .

Solution: If we recognize that is antiderivative of . We can evaluate the integral as

Example 32/ Find the following indefinite integrals, and check your answers by differentiation:

1- , 2- , 3- , 4- , 5-

Solution:

1-

2-

3-

4-

5-

H.W 1/ Find the indefinite integrals:

1- , 2- , 3- , 4- , 5-

H.W 2/ Verify the following formulas by differentiation:

1- , 2-

3- , 4- , 5-

H.W 3/ Solve the initial value problems in the following formulas:

1-

2-

3-

4-

5-

Definite integral:

Let be a function defined on a closed interval , we say that a number is the definite integral of over .

The symbol for the number in the definition of the definite integral is

Properties of definite integrals:

1-

2-

3-

4-

5-

Example 33/ Find the following definite integrals if:

1-

2-

3-

H.W 4/ Using properties of integrals and known values to find other integrals, suppose that are integrable and that equal:

Find the following integrals

Solution:

H.W 5/ Prove that

H.W 6/ Evaluate the following indefinite integrals by using the given substitutions to reduce the integrals to standard form.

Solution:

1-

Example 34/ Evaluate the following definite integrals by using the given substitutions to reduce the integrals to standard form.

Solution:

method1:

=

Method2: transform the integral as an indefinite integral change back to ,
and use the original .

H.W 7/ Evaluate the following definite integrals by using the given substitutions to reduce the integrals to standard form.

Note:

a-

b-

Example 35/

Example 36/

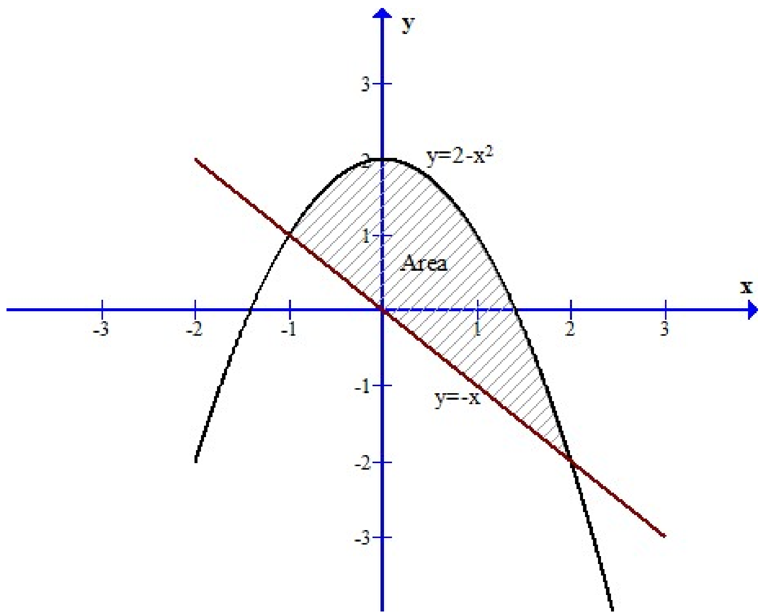
<u>Area between curves:</u>

If are continuous with throughout , then the area of the region between the curves from is the integral of .

Example 37/ Find the area of the region enclosed by the parabola .

Solution: The limits of integration are found by solving .

The region runs from , and the limits of integration are .

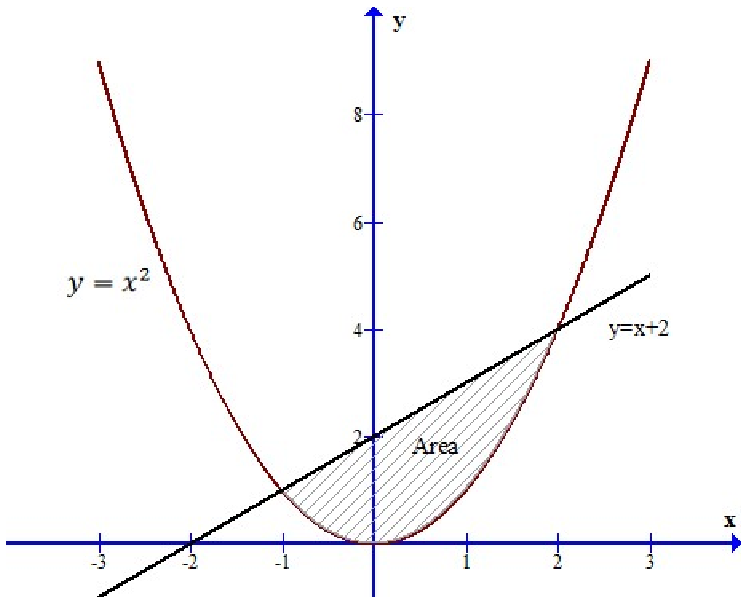


The area between curves is

Example 38/ Find the area of the region enclosed by the parabola .

Solution: The limits of integration are found by solving .

The region runs from , and the limits of integration are .



The area between curves is

<u>Area under a curve by integration:</u>

Case1- Curves which are entirely above the :

In this case, we find the area by simply finding the integral

Where the area under the curve from

Case2- Curves which are entirely below the :

In this case, the integral gives a negative number, we need to take the absolute value of this to find our area.

Case3- Part of the curve is below the , part of it is above the :

In this case, we have to sum the individual parts, taking the absolute value for the section where the curve is below the .

If we don’t do it like this, the "negative" area (the part below the) will be subtract from the "positive" part, and our total area will not be correct.

Case4- Certain Curves are much easier to sum vertically:

In some cases, it is easier to find the area if we take vertical sums. Sometimes the only possible way is to sum. We need to re-express this as , and we need to sum from bottom to top. So,

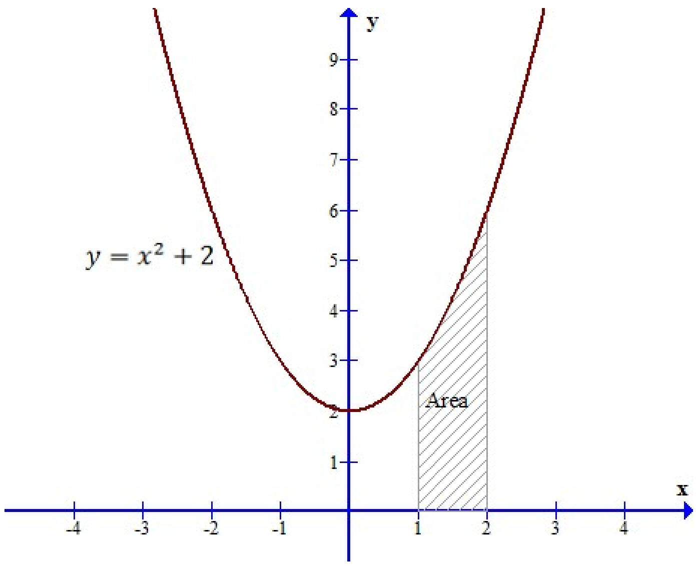
<u>Area between two curves using integration:</u>

Area bounded by the curves and the lines . We see that if we subtract the area under lower curve , from the area under the upper curve , then we will find the required area. This can be achieved in one step

Likewise, we can sum vertically by re-expressing both functions so that they are functions of , and we find

Where, as the limits on the integral.

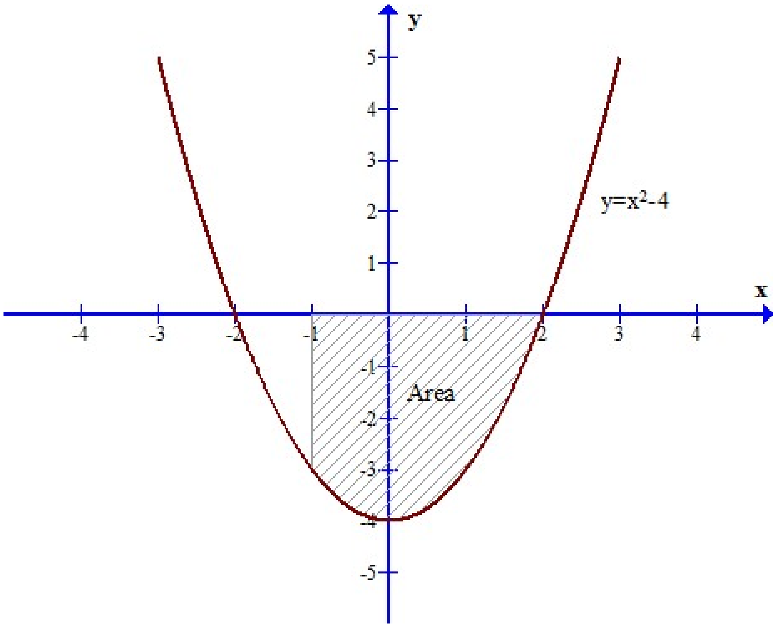
Example39/ Find the area underneath the curve from .



Solution:

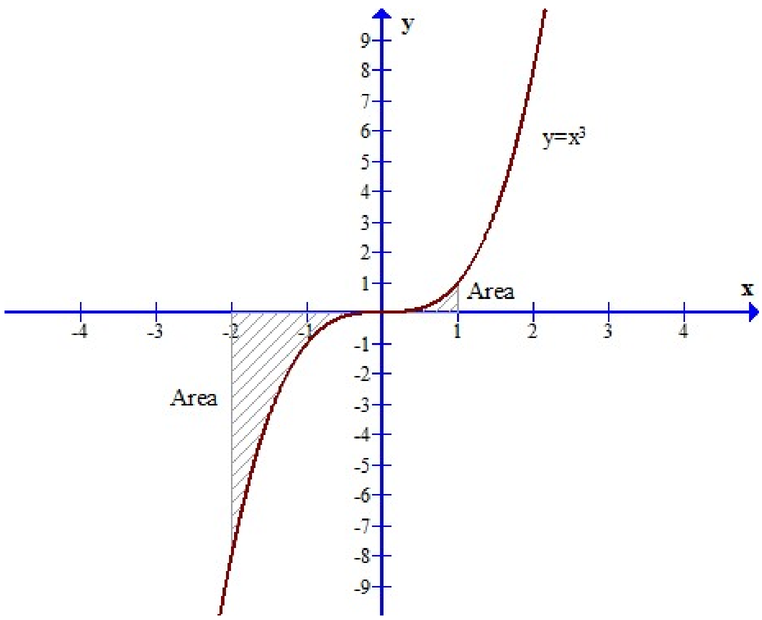
Example40/ Find the area bounded by , the axis and the lines .

Solution:



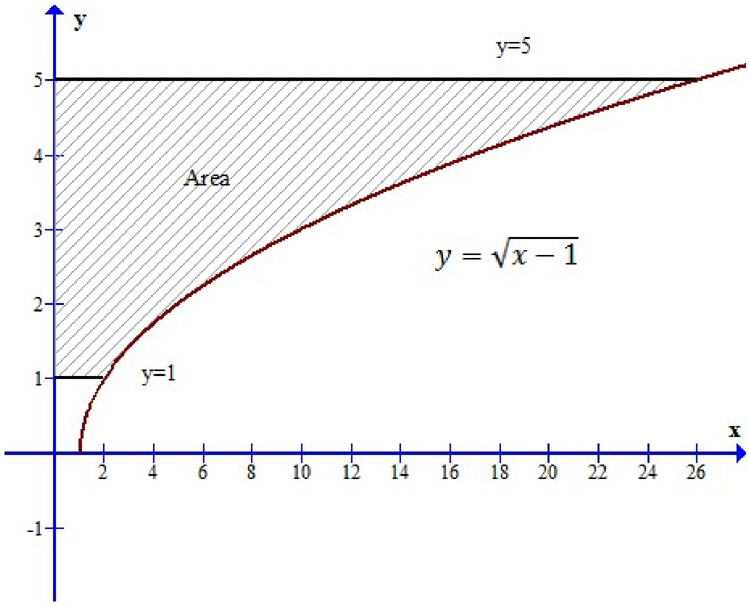
Example41/ What the area bounded by the curve and the lines .

Solution:



Example42/Find the area of the region bounded by the curve .

Solution:



In this case, we express :

So, the area is given by:

Example43/ Find the area bounded by .

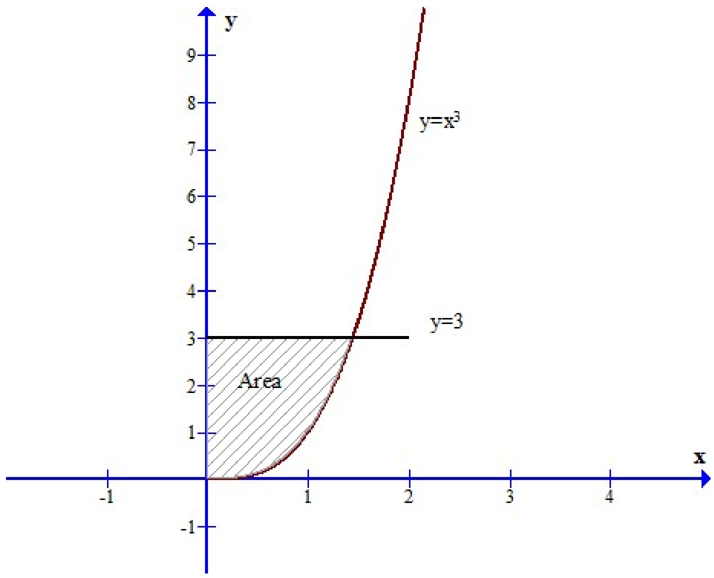
Solution: we must convert

The area equal

Or another method : We substitute

And the integral limitations equal .

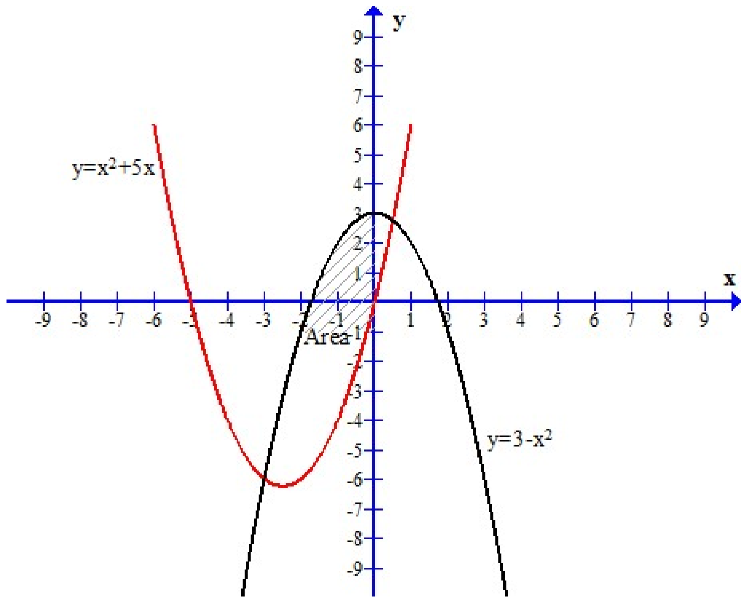
Then , we can use horizontally method to find area between the above function and the below function .



The area equal

Example44/ Find the area between the curves .

Solution:



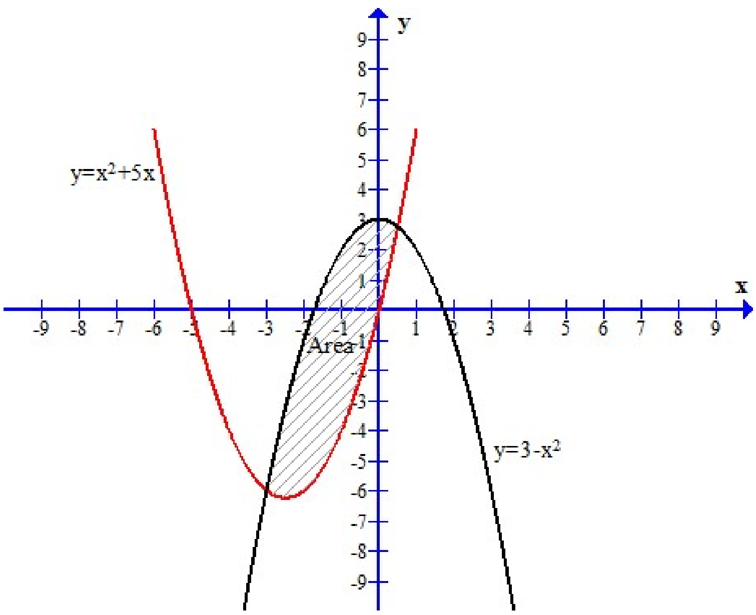
Note/ some of the shaded area is above the and some of it is below, therefore don't worry about taking absolute value, where the formula takes care of that automatically.

Note/ we can take any from the functions if it below or above , and if we get on the negative value of area , then we can take the absolute value of it.

Note/ if in question gave you only two functions for the curves, then from equating them, you get on the integral limitations .

Example45/ Find the area between the curves .

Solution: by equating the two function to get on the integral limitations



Volume of Solid:

Many solid objects, especially those made on a lathe have a circular cross-section and curved sides. In this section, we see how to find the volume of such objects using integration.

Example46/ Consider the area bounded by the straight line .

Solution: when the shaded is rotated about the , a volume is generated, and the resulting solid is a Cone.

To find this volume, we could take slices

The typical disk with dimensions, , where the volume of Cylinder is given by , and because

Adding the volumes of the disks (with infinitely small , we obtain the following formula:

, which means

Where, is the equation of the curve whose area is being rotated. are limits of the area being rotated. shows that the area is being rotated about the .

By applying volume formula to the earlier example, we have:

And we can find the volume of the cone using the following:

(Check ok.)

Example47/ Find the volume if the area bounded by the curve , and the limits of is rotated around .

And the area can be finding by the following:

<u>Volume by rotating the area enclosed between two curves:</u>

If we have two curves that enclose some area and we rotate that area around the , then the volume of the solid formed is given by

Where the limits for the region indicated by the vertical lines at . , represent lower and upper functions, respectively.

Example48/ A cup is made by rotating the area between functions around the . Find the volume of the material needed to make the cup. Units are .

Solution: to find integration limits, we equate the two functions, then we get

Rotation around the y-axis:

When the shaded area is rotated about the , the volume that is generated can be found by:

, which means

Where: is the equation of the curve expressed in terms of .

are the upper and lower limits of the area being rotated.

shows that the area is being rotated about the .

Example49/ Find the volume of the solid of revolution generated by rotating the curve .

Solution:

Example50/ Find the volume of the solid of revolution generated by rotating the curve .

Solution: we can find the volume by two methods, in this case take about , therefore

Arc Length of a Curve by Using Integration:

If the horizontal distance is (a small change in) and the vertical height of the triangle is (a small change in), then the length of the curved is approximated as:

And this equation represents general form of the length of the curve.

The length of the curve is given by:

Note/ we are assuming the function is continuous in the region (otherwise, the formula won't work).

Arc Length by Using Radian Measurement:

In this section, we see some of the common applications of radian measure, including arc length, area of sector of a circle, and angular velocity.

The length , of an of a circle radius subtended by (in radians) is given by:

If is in meters, will also be in meters.

Example51/ Find the length of the of a circle with radius and central angle .

Solution:

Area of Sector:

The area of a sector with central angle (in radians) is given by:

If is measured in , the area will be in .

Example 52/ Find the area of the sector with radius and central angle .

Solution:

Angular velocity: The time rate of change of angle by a rotating body is the angular velocity, written . It is measured in .

If is a linear velocity in and is the radius of the circle in , then

Example53/ A bicycle with tyres in diameter is travelling at . What is the angular velocity of the tyre in .

Solution: we convert the units to

<u>Finite sums:</u>

Sigma notation enables us to write a sum with many terms in the compact form

The index k ends at k=n

Summation symbol is a formula for the term

Index k start at k=1

Example 36/

H.W 8/ Write the sums of the following notations:

1- , 2- , 3- , 4-

5- , 6- , 7-

Solution :

7-

<u>Algebra rules for finite sums:</u>

1- Sum rules:

2- Difference rule:

3- Constant multiple rule:

4- Constant value rule: , Ex/